Exercise 27

Use a graph of the vector field $\mathbf{F}$ and the curve C to guess whether the line integral of $\mathbf{F}$ over C is positive, negative, or zero. Then evaluate the line integral.

$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$,

C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

Solution

The parameterization for a circular arc going from $(2, 0)$ to $(0, -2)$ in a counterclockwise fashion is $x(t) = 2\cos t$ and $y(t) = 2\sin t$ with $0 \leq t \leq 3\pi/2$. As a result, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^{3\pi/2} \langle x(t) - y(t), x(t)y(t) \rangle \cdot \frac{d}{dt}\langle 2\cos t, 2\sin t\rangle\,dt \\
&= \int_0^{3\pi/2} \langle (2\cos t) - (2\sin t), (2\cos t)(2\sin t) \rangle \cdot \langle -2\sin t, 2\cos t\rangle\,dt \\
&= \int_0^{3\pi/2} [(2\cos t - 2\sin t)(-2\sin t) + (2\cos t)(2\sin t)(2\cos t)]\,dt \\
&= \int_0^{3\pi/2} (-4\cos t\sin t + 4\sin^2 t + 8\cos^2 t\sin t)\,dt \\
&= -4\int_0^{3\pi/2} \cos t\sin t\,dt + 4\int_0^{3\pi/2} \sin^2 t\,dt + 8\int_0^{3\pi/2} \cos^2 t\sin t\,dt \\
&= -4\int_0^{3\pi/2} \frac{1}{2}\sin 2t\,dt + 4\int_0^{3\pi/2} \frac{1}{2}(1 - \cos 2t)\,dt + 8\int_0^{3\pi/2} \cos^2 t\sin t\,dt \\
&= -2\int_0^{3\pi/2} \sin 2t\,dt + 2\int_0^{3\pi/2} (1 - \cos 2t)\,dt + 8\int_0^{3\pi/2} \cos^2 t\sin t\,dt.
\end{aligned}$$

Make the following substitution in the third integral.

$$\begin{aligned}
u &= \cos t \\
du &= -\sin t\,dt \quad \rightarrow \quad -du = \sin t\,dt
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= -2\int_0^{3\pi/2} \sin 2t\, dt + 2\int_0^{3\pi/2} (1-\cos 2t)\, dt + 8\int_{\cos(0)}^{\cos(3\pi/2)} u^2(-du) \\ &= -2\left(-\frac{1}{2}\cos 2t\right)\Big|_0^{3\pi/2} + 2\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{3\pi/2} + 8\int_0^1 u^2\, du \\ &= (\cos 3\pi - \cos 0) + 2\left(\frac{3\pi}{2}\right) + 8\left(\frac{1}{3}\right) \\ &= (-2) + (3\pi) + \frac{8}{3} \\ &= 3\pi + \frac{2}{3}.\end{aligned}$$

A plot of the vector field $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$ is shown below.